

Polynomials

Resultants

Resultant of Two Polynomials

$$Q(X) = AX^2 + BX + C$$

$$R(X) = DX^2 + EX + F$$

$$\mathbf{R}(Q, R) = f(A, B, C, D, E, F)$$

$\mathbf{R} = 0 \hat{\cup} Q$ and R have a common root

Calculating the Resultant

$$Q(X) = AX^2 + 2BX + C = 0$$

$$R(X) = DX^2 + 2EX + F = 0$$

$$aQ + bR = 0$$

$$DQ - AR = 0$$

$$D(AX^2 + 2BX + C)$$

$$- A(DX^2 + 2EX + F) = 0$$

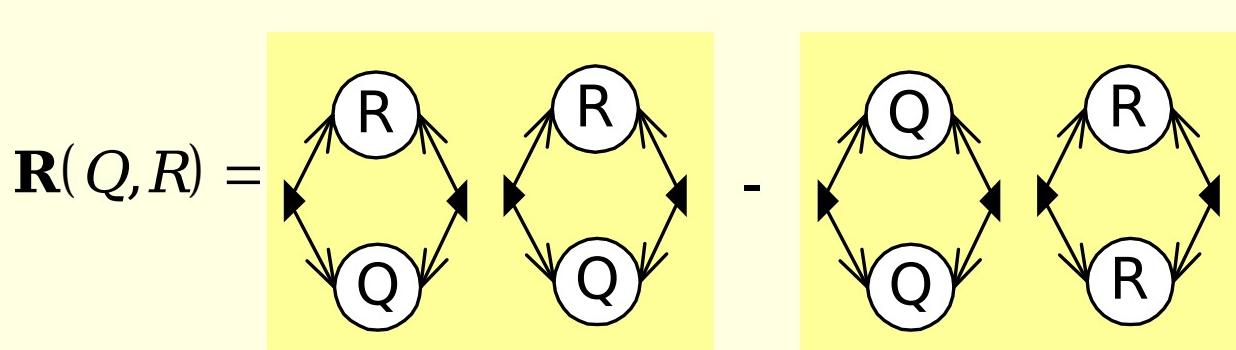
$$2(BD - AE)X + (CD - AF) = 0$$

Resultant of Q and R

$$Q(x, w) = Ax^2 + 2Bxw + Cw^2$$

$$R(x, w) = Dx^2 + 2Exw + Fw^2$$

$$\begin{aligned} \mathbf{R}(Q, R) = & +A^2F^2 - 4ABEF + 4ACE^2 - 2ACDF \\ & + 4B^2DF - 4BCED + C^2D^2 \end{aligned}$$

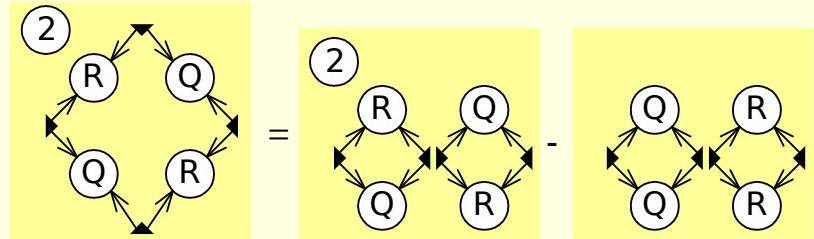
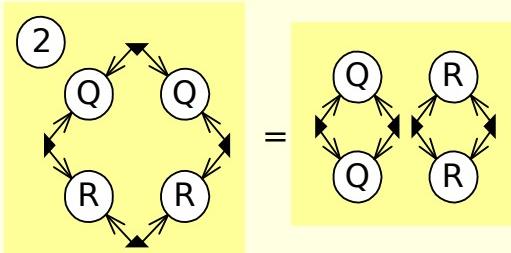


Two Forms of Resultant

$$\mathbf{R}(Q, R) = \text{Diagram A} - \text{Diagram B}$$

$$\mathbf{R}(Q,R) = \begin{array}{c} \text{Diagram A} \\ \text{Diagram B} \end{array} - \begin{array}{c} \text{Diagram C} \\ \text{Diagram D} \end{array}$$

Identities:

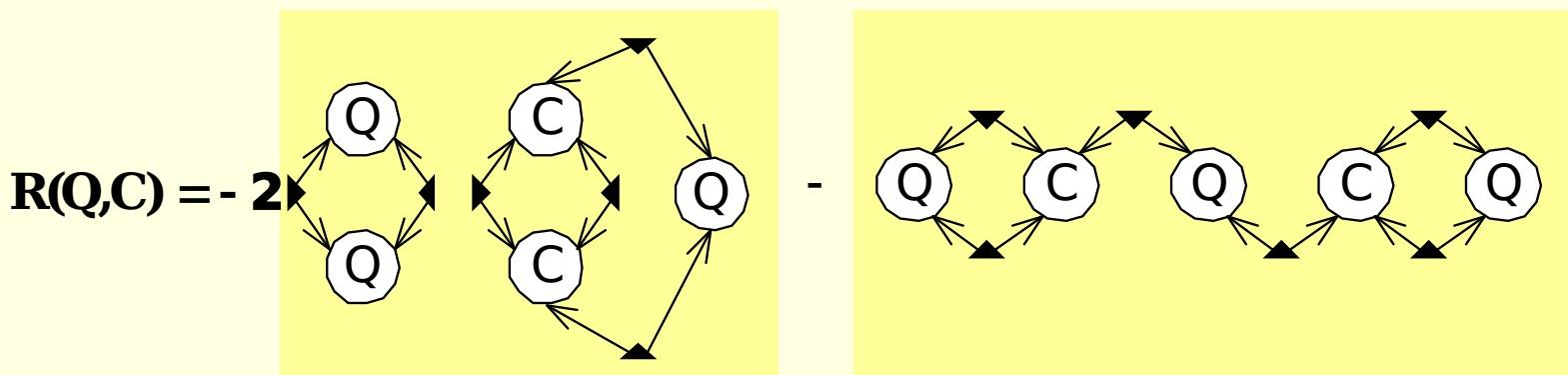


Resultant of Q and C

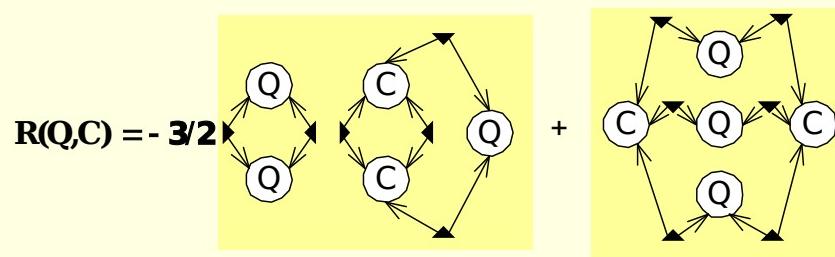
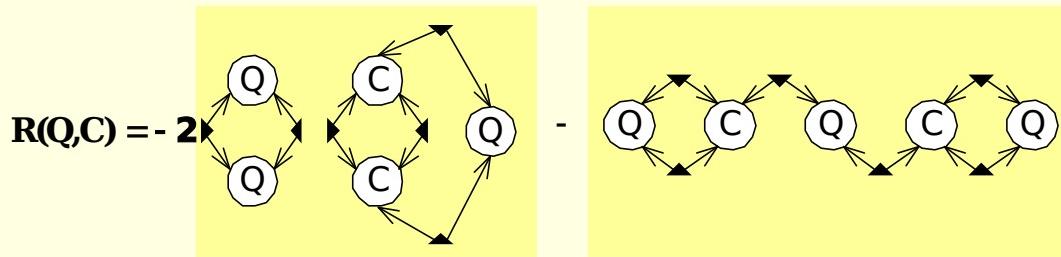
$$C(x, w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

$$Q(x, w) = Ex^2 + 2Fxw + Gw^2$$

$$\begin{aligned} R(Q, C) = & -A^2(G^3) + 6AB(FG^2) - 6AC(2F^2G - EG^2) - 2AD(-4F^3 + 3EFG) \\ & - 9B^2(EG^2) + 18BC(EFG) - 6BD(2EF^2 - E^2G) \\ & - 9C^2(E^2G) + 6CD(E^2F) \\ & - D^2(E^3) \end{aligned}$$

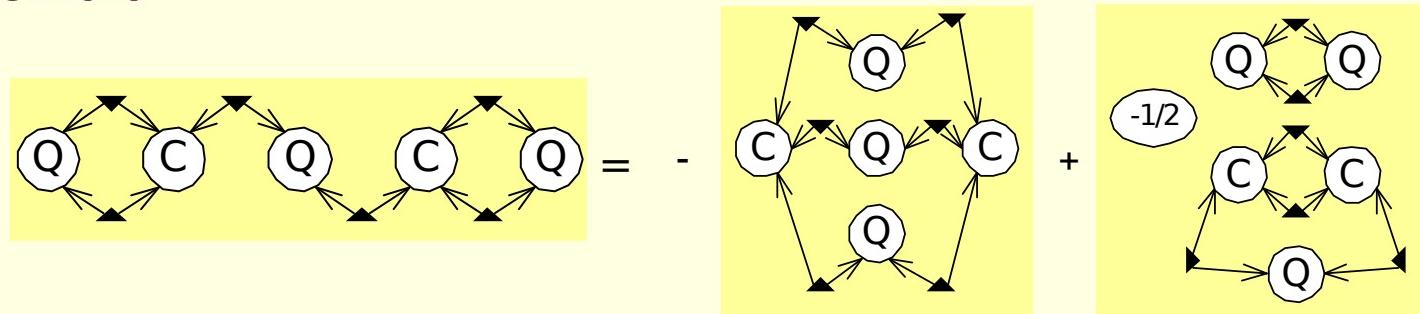


Forms of Q,C Resultant

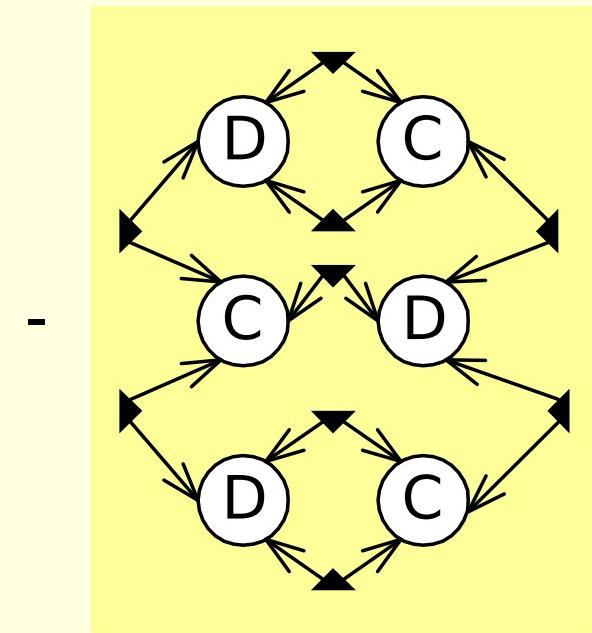
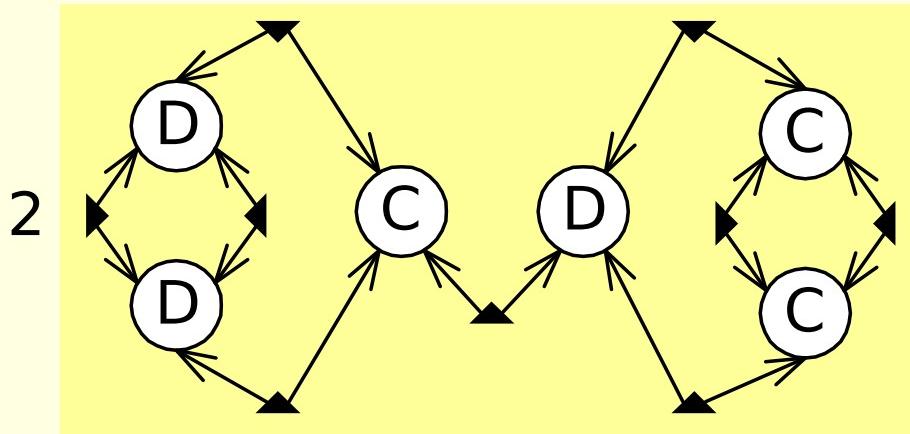


Identit

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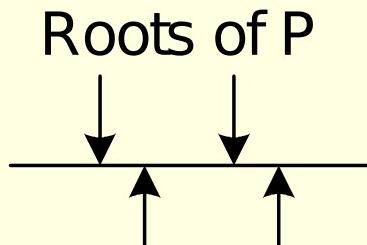


Resultant of two Cubics ?

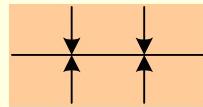
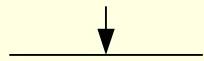
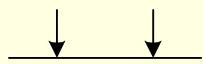
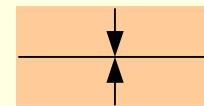
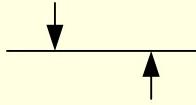
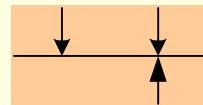
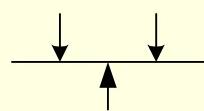
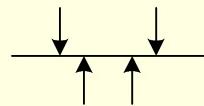
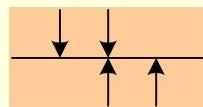
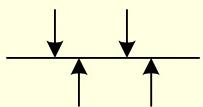


What's Really Going on With Resultants?

Possible Relationships between P and Q

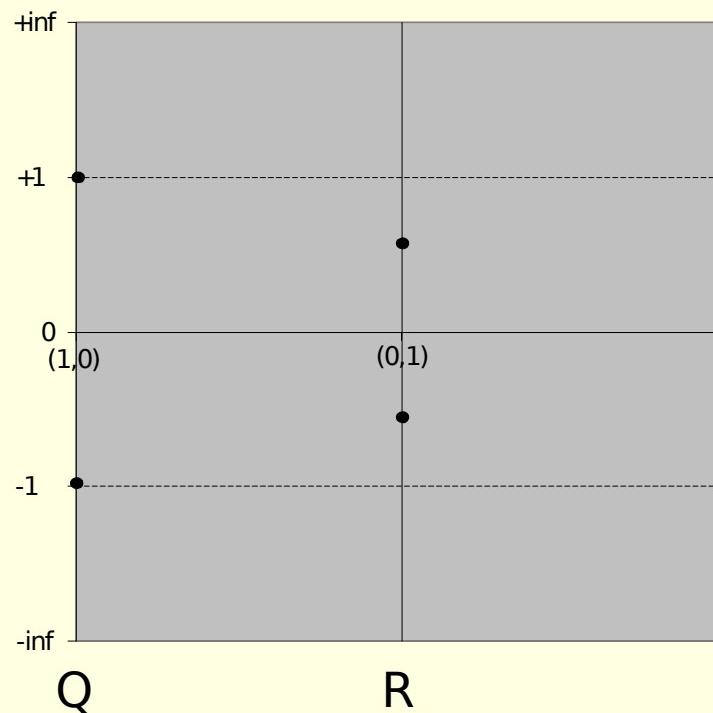


Roots of Q



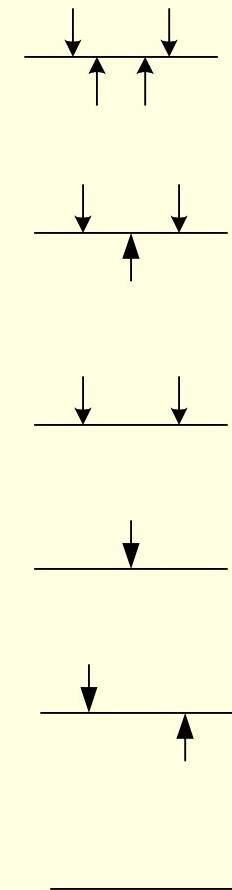
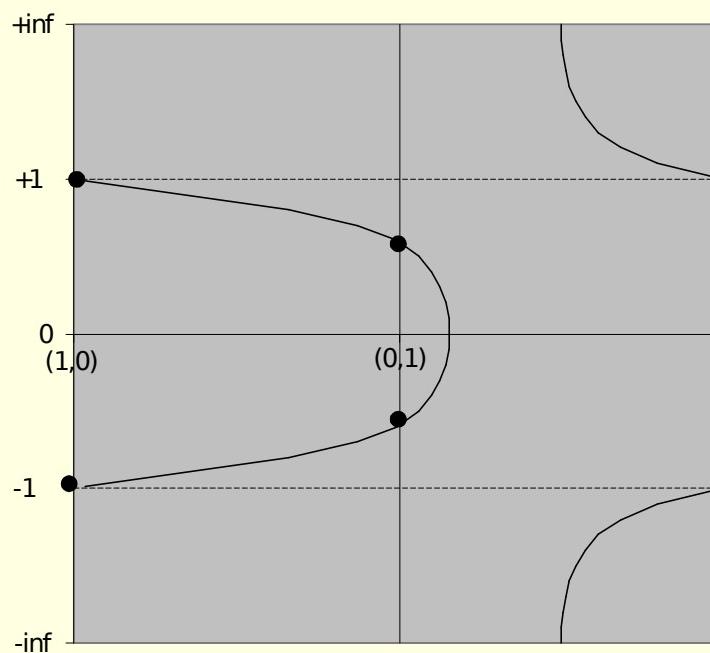
Linear Combos of Q and R

$$aQ + bR = 0$$



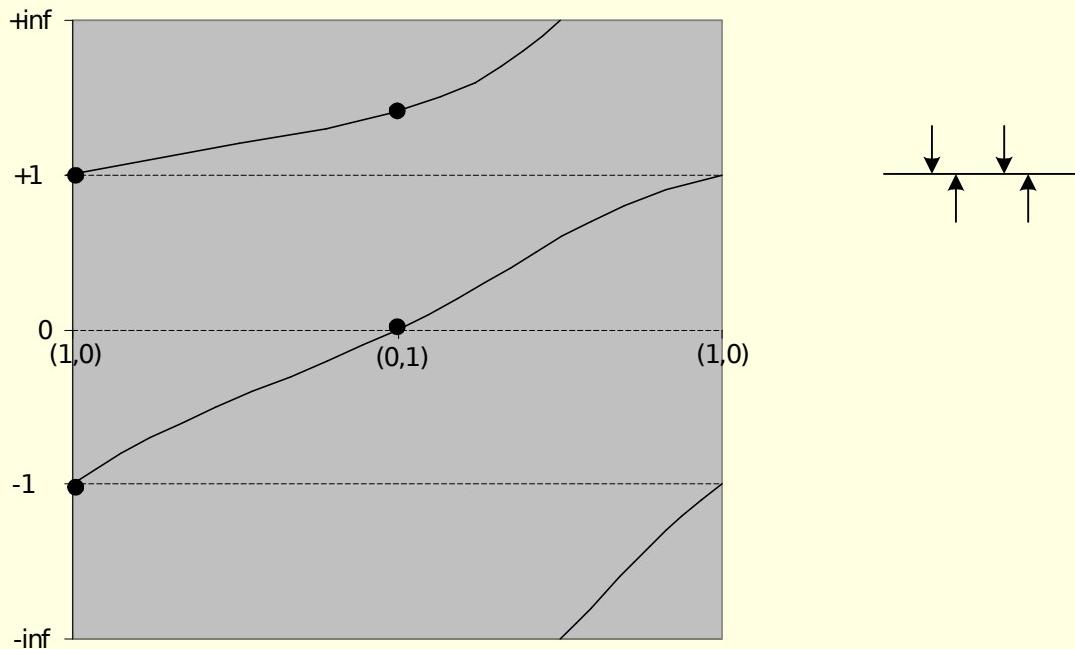
Q,R Have Enclosed Roots

$$aQ + bR = 0$$



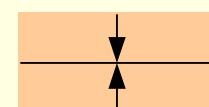
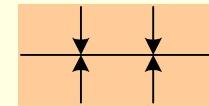
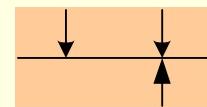
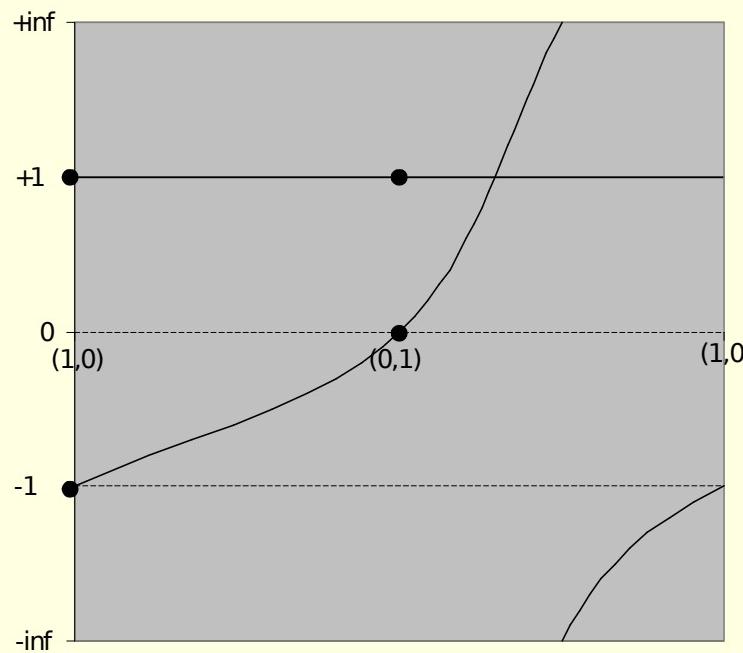
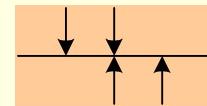
Q,R Have Interleaved Roots

$$aQ + bR = 0$$

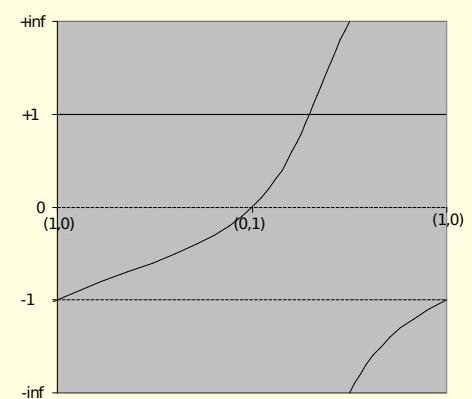
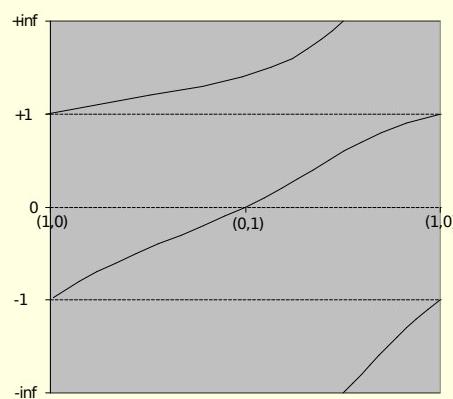
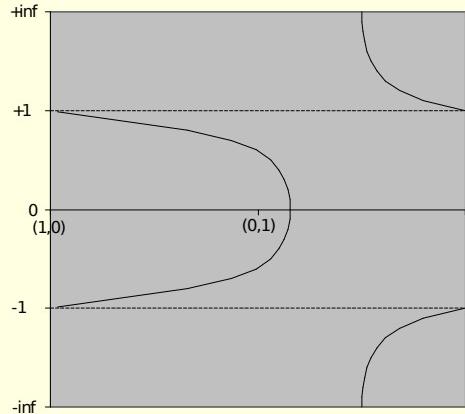


Q,R Have Common Root

$$aQ + bR = 0$$



Three Possible Evolutions of Roots of $aQ+bR$

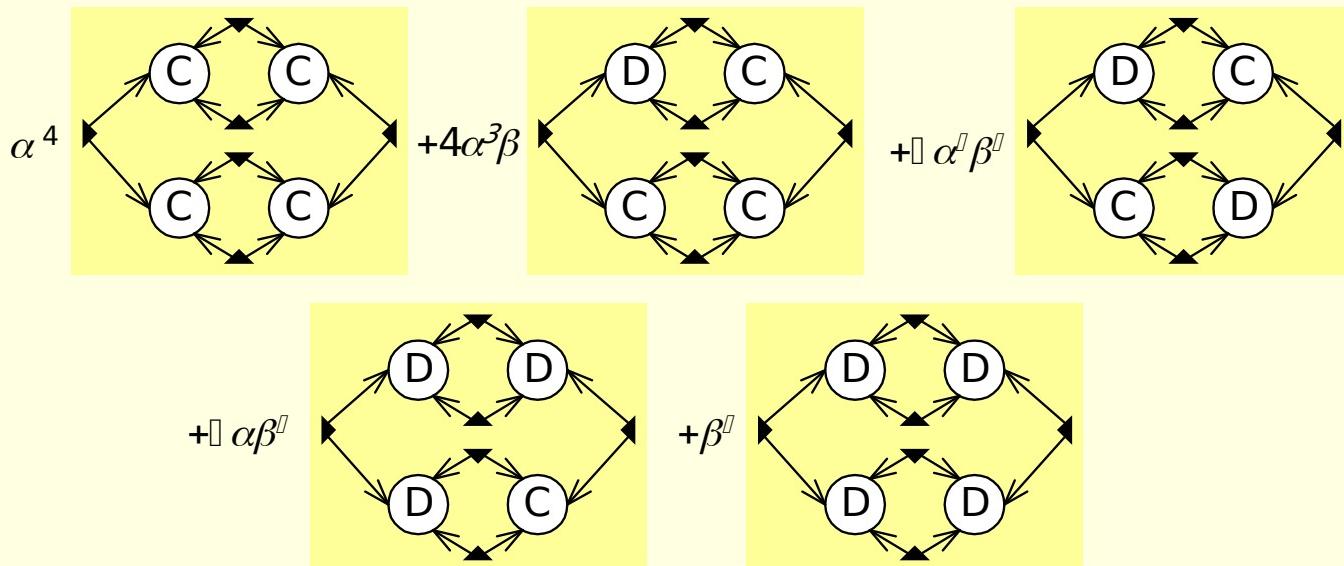


$$\det(aQ + bR) = \alpha^2 \begin{array}{c} \text{Diagram: } Q \text{ and } R \text{ nodes in a circle, } Q \text{ has two arrows to } R, R \text{ has one arrow to } Q \\ \text{Diagram: } Q \text{ and } R \text{ nodes in a circle, } R \text{ has two arrows to } Q, Q \text{ has one arrow to } R \end{array} + 2\alpha\beta \begin{array}{c} \text{Diagram: } Q \text{ and } R \text{ nodes in a circle, } Q \text{ has two arrows to } R, R \text{ has two arrows to } Q \\ \text{Diagram: } Q \text{ and } R \text{ nodes in a circle, } R \text{ has two arrows to } Q, Q \text{ has two arrows to } R \end{array} + \beta^2 \begin{array}{c} \text{Diagram: } R \text{ node only} \\ \text{Diagram: } Q \text{ node only} \end{array}$$

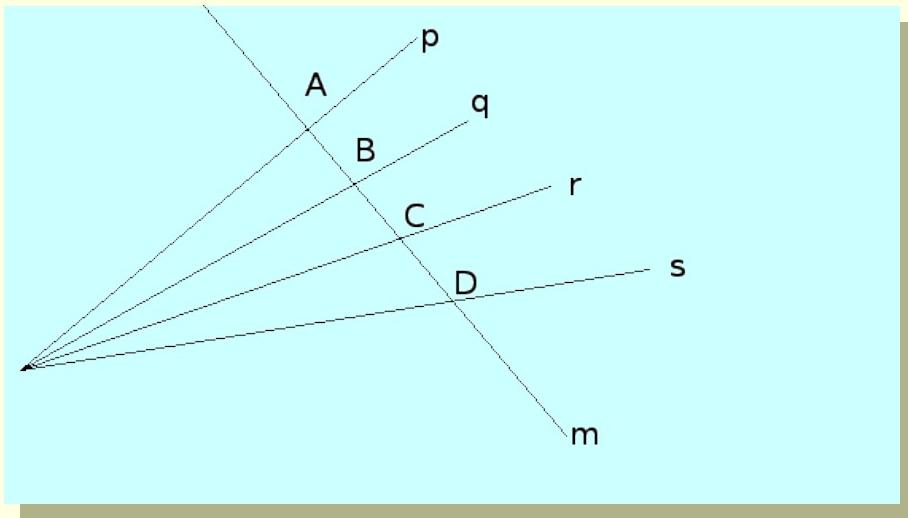
$$\mathbf{R}(Q, R) = \begin{array}{c} \text{Diagram: } R \text{ and } Q \text{ nodes in a circle, } R \text{ has two arrows to } Q, Q \text{ has one arrow to } R \\ \text{Diagram: } R \text{ and } Q \text{ nodes in a circle, } Q \text{ has two arrows to } R, R \text{ has one arrow to } Q \end{array} - \begin{array}{c} \text{Diagram: } Q \text{ and } R \text{ nodes in a circle, } Q \text{ has two arrows to } R, R \text{ has two arrows to } Q \\ \text{Diagram: } Q \text{ and } R \text{ nodes in a circle, } R \text{ has two arrows to } Q, Q \text{ has two arrows to } R \end{array}$$

Possible Evolutions of Roots of Two Cubics

$$\det(a\mathbf{C} + b\mathbf{D}) =$$



The Cross Ratio



$$\chi = \frac{|AB|/|BD|}{|AC|/|CD|}$$

$$c = \frac{(p'q)m)(r's)m)}{(q's)m)(p'r)m)}$$

Generalized Cross Ratio of Two Quadratic Polynomials

$$\chi = \frac{\text{Top Diagram}}{\text{Bottom Diagram}}$$

The equation $\chi = \frac{\text{Top Diagram}}{\text{Bottom Diagram}}$ represents the generalized cross ratio of two quadratic polynomials. The top diagram shows a configuration where R is at the top-left and top-right positions, and Q is at the bottom-left and bottom-right positions. The bottom diagram shows a configuration where Q is at the top-left and top-right positions, and R is at the bottom-left and bottom-right positions. Both diagrams feature a square-like arrangement of four nodes with clockwise circular arrows connecting them.